Parallel curve of a Clothoid *

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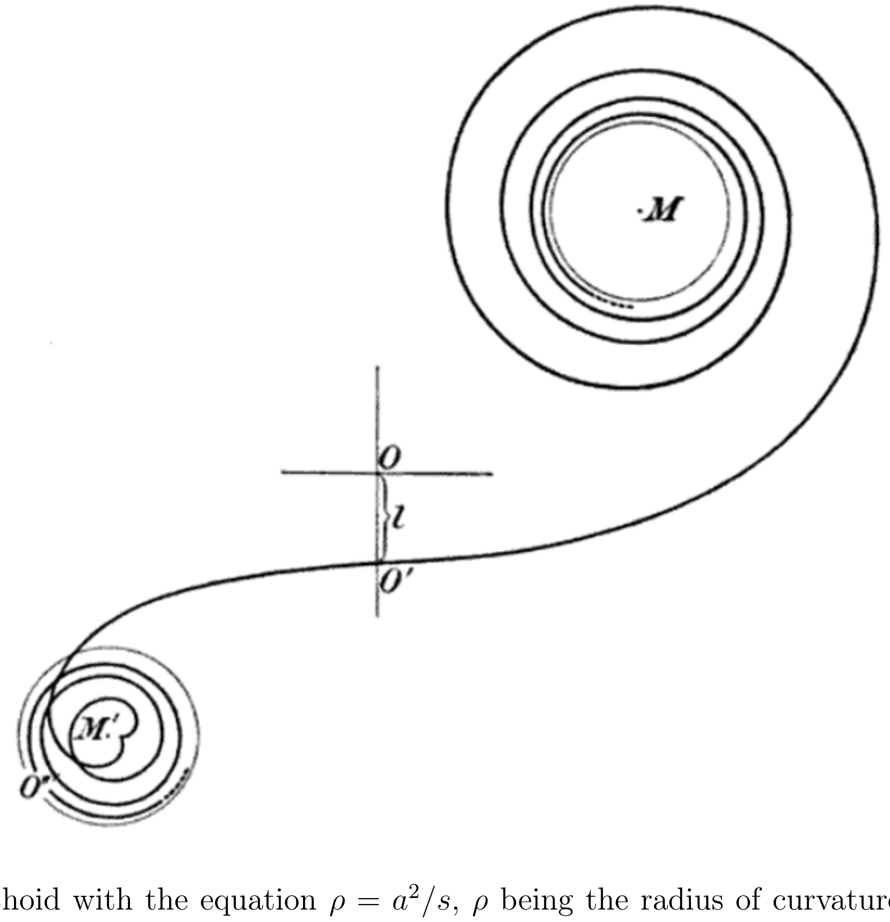
Figure 1: Parallel curve of a clothoid



The clothoid with the equation $\rho = a^2/s$, ρ being the radius of curvature and s the arc, has an inflection point $[s = \pm 0, \rho = \pm\infty]$ in its starting point O. Additionally it has two asymptotic points M and M' (Tangent angle $\varphi = s^2/2a^2$, which is ∞ for $s = \pm\infty; \rho = 0$). Their coordinates are $x = y = \pm\frac{a}{2}\sqrt{\pi}$.

*Translated from the german original, found in the 1907 edition of the *Archiv der Mathematik und Physik*

Its parallel curve does not seem to have been examined yet. For it

$$s' = s + l\varphi, \rho' = \rho + l \tag{1}$$

applies, l being the distance (Cesàro, *Nat. G.*, §19. — Loria, S. 645). This immediately results in

$$s = \frac{a^2}{\rho' - l}, s' = s + \frac{s^2 l}{2a^2}$$

and by insertion

$$s' = \frac{a^2(2\rho' - l)}{2(\rho' - l)^2} \tag{2}$$

or transposed

$$\rho' = \frac{a^2}{2s'} + l \pm \frac{a}{2s'}\sqrt{a^2 + 2ls'} \tag{2*}$$

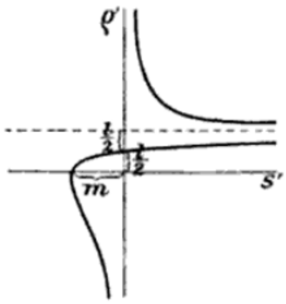
. Further

$$\varphi' = \int \frac{ds'}{\rho'} = \int \frac{\left(1 + \frac{ls}{a^2}\right) ds}{\frac{a^2}{s} + l} = \int \frac{sds}{a^2} = \varphi \tag{3}$$

as is obvious for a parallel curve.

From (2) and (2*), s' is a distinct function of ρ' while ρ' is an ambiguous function of s'. To better see this interdependency, drawing (2) into a right-angled coordinate-system (see figure 2) helps.

Figure 2: Relation of s' and ρ'



For $s' = \pm 0$, ρ will be $\pm\infty$; this yields the inflection point in the starting point O' of the arc. If we turn right, ρ' falls with rising s', up to l because for $s' = +\infty$, ρ' equals l. But, because of (3), φ' is also ∞. This means, the curve has an asymptotic circle around M with radius l, which it closes in on from the outside. If we now go left from O', ie we take s' negative, then ρ' goes from $-\infty$ to 0 at $s' = -m = -a^2/2l^2$. Because of (2*) this is also a minimum for s'. Our curve has a sharp turn at this point, that we will call S, with the tangential direction $\varphi' = a^2/2l^2$. The smaller l is in relation to the clothoid, the more turns around M' the branch going left from O' will make until it reaches S. From S onwards s' rises (as seen in figure 2) up to the point O'' for which $|O'S| = |SO''|$. In O'', s' is 0 and $\rho' = \frac{1}{2}l$. Apart from that, the point is not visible on the curve. Continuing on, s' and φ' go to $+\infty$ while ρ' tends to l. This means, the curve has a second asymptotic circle around M', which it closes in on from the inside.

Changing the sign of l just switches M and M'.